Exactly Soluble System (II)
(b)

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SEPARATION OF VARIABLES

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$
$$+\frac{8\pi^2}{h^2}\mu(E-V)\psi=0 \quad -(1)$$

Equation(1) is a Second order Partial differential equation. This con: three variables. In order to separate the variables it becomes necessary to assume that ψ may be represented by the Product of three wave functions, each having only one of the three variables, r,θ and ϕ. If we let

$$\psi = R_{(r)}\, T_\theta\, F\phi$$

and make this substitution into equation (1) we get

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(r^2\frac{\partial}{\partial r}R_{(r)}T_\theta F\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}R_{(r)}T_{(\theta)}F_{(\phi)}$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}R_{(r)}T_{(\theta)}F_{(\phi)}\right]+\frac{8\pi^2\mu}{h^2}[E-V]R_{(r)}T_{(\theta)}F_{(\phi)}=0$$

Which on dividing by $R_{(r)}\,T_{(\theta)}\,F_{(\phi)}$ gives.

$$\frac{1}{r^2}\frac{1}{R_{(r)}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{(r)}}{\partial r}\right)+\frac{1}{F_{(\phi)}r^2\sin^2\theta}\frac{\partial^2F\phi}{\partial\phi^2}$$
$$+\frac{1}{T_{(\theta)}r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_{(\theta)}}{\partial\theta}\right)+\frac{8\pi^2\mu}{h^2}[E-V]=0$$

If we multiply by $r^2\sin^2\theta$, we get

$$\frac{\sin^2\theta}{R_{(r)}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{(r)}}{\partial r}\right)+\frac{1}{F_{(\phi)}}\frac{\partial^2F_{(\phi)}}{\partial\phi^2}+\frac{\sin\theta}{T_{(\theta)}}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_\theta}{\partial\theta}\right)$$
$$+\frac{8\pi^2\mu r^2\sin^2\theta}{h^2}(E-V)=-\frac{1}{F_{(\phi)}}\frac{\partial^2F_{(\phi)}}{\partial\phi^2} \quad -(2)$$

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The left side of equation (2) has only the variables r and θ, whereas the right side of the equation has only the variable ϕ. If we put the right side equal to m^2, equation (2) modifies to

$$\frac{\sin^2\theta}{R(r)}\frac{\delta}{\partial r}\left(r^2\frac{\delta R(r)}{\partial r}\right)+\frac{\sin\theta}{T(\theta)}\frac{\delta}{\partial\theta}\left(\sin\theta\frac{\delta T_\theta}{\partial\theta}\right)+\frac{8\pi^2\mu r^2\sin\theta}{h^2}\cdot(E-V)=m^2 \quad -(3)$$

where
$$\frac{1}{f(\phi)}\frac{\delta^2 f(\phi)}{\partial\phi^2}=-m^2 \quad — (4)$$

Equation (3) contains two variables r and θ. The problem now is to carry out the separation of the remaining two variables r and θ. By dividing equation (3) by $\sin^2\theta$, we get

$$\frac{1}{R(r)}\frac{\delta}{\partial r}\left(r^2\frac{\delta R(r)}{\partial r}\right)+\frac{1}{T(\theta)\sin\theta}\frac{\delta}{\partial\theta}\left(\sin\theta\cdot\frac{\delta T(\theta)}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}+\frac{8\pi^2\mu r^2}{h^2}(E-V)=0.$$

Or on rearranging

$$\frac{1}{R(r)}\frac{\delta}{\partial(r)}\left(r^2\frac{\delta R(r)}{\partial r}\right)+\frac{8\pi^2\mu r^2}{h^2}[E-V)=\frac{m^2}{\sin^2\theta}-\frac{1}{T(\theta)\sin\theta}\frac{\delta}{\partial\theta}\left(\sin\theta\frac{\delta T(\theta)}{\partial\theta}\right) \quad -(5)$$

As each side of the equation (5) contains only one variable, they both must be equal to the same constant. If the right side of the equation is equal to the constant β and this gives on multiplication by $T(\theta)$

$$\frac{m^2 T(\theta)}{\sin^2\theta} - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dT(\theta)}{d\theta}\right) - \beta T(\theta) = 0 \quad -(6)$$

This is the desired form of the T equation. The remaining part of the original equation is the R equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR(r)}{dr}\right) - \frac{\beta}{r^2}R(r) + \frac{8\pi^2\mu}{h^2}[E-V]R(r) = 0 \quad -(7)$$

Thus, the three variables have been successfully separated and the three independent total differential equations that result are:

(i) $$\frac{d^2 f(\phi)}{d\phi^2} + m^2 f(\phi) = 0 \quad -(8)$$

(ii) $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dT(\theta)}{d\theta}\right) - \frac{m^2 T(\theta)}{\sin^2\theta} + \beta T(\theta) = 0 \quad -(9)$$

(iii) $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR(r)}{dr}\right) - \frac{\beta}{r^2}R(r) + \frac{8\pi^2\mu}{h^2}(E-V)R(r) = 0 \quad -(10)$$

Solution of $f(\phi)$ equation

$$\frac{d^2 f}{d\phi^2} + m^2 f(\phi) = 0$$ is the same form as the wave equation for the particle in a box. In terms of sin and cos.