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Exact Solution

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HERMITE POLYNOMIALS

Hermite Polynomials were defined by Pierresimon Laplace in 1810, though in scarcely recognizable from, and studied in detail by Pafnuty chebyshov in 1859, Chebyshov's work was overlooked, and they were named later after Charles Hermite, who wrote on the Polynomials in 1864, describing them as new. Hermite was the first to define the multidimensional Polynomials in his later 1865

Since $\lambda = (2n+1)$

$$H - 2\xi H' + (\lambda-1)H = 0 \quad - (1)$$

Equation(1) becomes

$$H_n'' - 2\xi H_n' + 2n H_n = 0 \quad (11)$$

The Polynomials $H_n(\xi)$ are called the Hermite Polynomials. with a convenient formulation $H_n(\xi)$ Can be expressed in terms of a generating function ξ.

$S(\xi, s)$.

$$S(\xi, s) = e^{+\xi^2 - (s-\xi)^2} = e^{-s^2+2s\xi}$$

$$= 1 - s^2 + 2s\xi + \frac{(s^2+2\ )^2}{\underline{|2}} + \cdots$$

$$= 1 + 2s\xi + (4\xi^2 - 2)\frac{s^2}{\underline{|2}} + -$$

$$= H_0 + H_1 s + H_2 \frac{s^2}{\underline{|2}} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{H_n(\xi)s^n}{\underline{|n}} \quad \text{———}$$

Hence

$$e^{-s^2+2s\xi} = \sum_{n=0}^{\infty} \frac{H_n(\xi)s^n}{\underline{|n}} \quad - (111)$$

To show that $H_n(\xi)$ satisfies the differential equation (1) first with respect to ξ and than with respect to s.

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$$\frac{\delta S}{\delta \xi} = 2s e^{-s+2s\xi} = \Sigma_n \frac{s^n}{\lfloor n} H_n' \xi$$

or $$\sum_n \frac{2s^{n+1}}{\lfloor n} H_n(\xi) = \Sigma \frac{s^n}{\lfloor n} H_n'(\xi)$$

$$\frac{\partial S}{\partial s} = (-2s + 2\xi) e^{-s^2+2s\xi} = \sum_n \frac{s^{n-1}}{\lfloor(n-1)} H_n(\xi)$$

or $$\sum_n \frac{(-2s+2\xi)s^n}{\lfloor n} H_n \xi = \sum_n \frac{s^{n-1}}{(n-1)} H_n(\xi)$$

Equating equal Powers of s on both side of these two equations.

$$H_n' = 2n H_{n-1} \quad \text{— (IV)}$$

$$H_{n+1} = 2\xi H_n - 2n H_{n-1} \quad \text{—(V)}$$

differentiating equation (IV) & (V) with respect to ξ, we get

$$H_n'' = 2n H'_{n-1}$$

$$H'_{n+1} = 2\xi H_n' - 2n H'_{n-1} + 2H_n.$$

Adding these two, we get

$$H_n'' + H'_{n+1} = 2\xi H_n' + 2H_n \quad \text{—(VI)}$$

But $H'_{n+1} = 2(n+1) H_n$ from equation (VI), we get

$$H_n'' + 2n H_n + H_n = 2\xi H_n' + H_n$$

or $$H_n'' - 2\xi H_n' + 2n H_n = 0$$

Thus H_n satisfies equation (II), so $H_n(\xi)$ are Hermite Polynomials.

If $S(\xi; s)$ is differentiated n times with respect to s and s is then set equal to zero, equation (III) shows that the result is simply $H_n(\xi)$. Now for any funtion

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of the form $f(S-\xi)$. it is apparent that

$$\frac{\delta f}{\delta s} = -\frac{\partial f}{\partial \xi}$$

Now $$\frac{\delta^n}{\partial s^n}\left[S'(\xi, s)\right] = \frac{\delta^n}{\delta s^n}\left(e^{-s^2+2s\xi}\right)$$

$$= \frac{\partial^n}{\delta s^n}\left[e^{\xi^2-(s-\xi)^2}\right]$$

$$= e^{\xi^2}\frac{\delta^n}{\delta s^n}\left[e^{-s(s-\xi)^2}\right]$$

$$= (-1)^n e^{\xi^2}\frac{\delta n}{\delta \xi}\left(e^{-s(-s-\xi)^2}\right]$$

This gives an expression for the n^{th} Hermite Polynomial

$$H_n(\xi) = \lim_{s\to 0}\frac{\delta^2}{\delta s^2}\left[S(\xi, s)\right] = (-1)^n e^{\xi^2}\frac{\delta^n}{\delta \xi^n}e^{-(\xi)^2} \quad \text{— (11)}$$

The first few Hermite Polynomials are

(1) $H_0(\xi) = 1$ (2) $H_1(\xi) = 2(\xi)$ (3) $H_2(\xi) = 4\xi^2 - 2$

(4) $H_3(\xi) = 8\xi^3 - 12\xi$ (5) $H_4(\xi) = 16\xi^4 - 48\xi + 12$

(6) $H_6(\xi) = 64x^6 - 480x^4 + 720x^2 - 120$

(7) $H_7(\xi) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

(8) $H_8(\xi) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$

(9) $H_9(\xi) = 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$

(10) $H_{10}(\xi) = 1024x^{10} = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$

(11) $H_5(\xi) = 32x^5 - 160x^3 + 120x,$

The n^{th}- order Hermite Polynomial is a Polynomial of degree n, H_n has leading Coefficient 2^n

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